

On the classification of metric hypercomplex group alternative-elastic algebras for n=8.

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In this article, the clarification to Note 4 [5] for n=8 is considered. In this connection, answers to the following questions are given.

1. How to classify the metric hypercomplex orthogonal group alternative-elastic algebras for n=8?
2. How to associate the metric hypercomplex orthogonal group alternative-elastic algebra to the symmetric controlling spin-tensor for n=8?
3. How technically to construct the symmetric controlling spin-tensor for n=8?

Let *product* of elements of *hypercomplex group alternative-elastic algebra* \mathbb{A} [5],[1],[7], [2], [3] over the field \mathbb{R} and the vector space \mathbb{R}^8 be defined as: 1). $\exists! c = ab$; 2). $\exists! e : ae = ea = a$; 3). $\forall a \neq 0 \exists! a^{-1} : aa^{-1} = a^{-1}a = e$; 4). $\forall a, b : (aa)b - a(ab) = b(aa) - (ba)a$; 5). $\forall a, b : a(ba) = (ab)a$; 6). $\forall a, b, c : a(b+c) = ab+ac, (b+c)a = ba+ca$. Construct one of such the algebras. Suppose that the vector space \mathbb{R}^8 is equipped with the metric $(i, j, \dots, A, B, \dots = \overline{1}, 8)$

$$\langle a, b \rangle_e := \frac{1}{2}(a\bar{b} + b\bar{a}), \quad a = a_1e + \sum_{r=2}^8 a_re_r, \quad \bar{a} = a_1e - \sum_{r=2}^8 a_re_r. \quad (1)$$

Then according to the equation (5) [5]

$$\eta_{ij}{}^k := \sqrt{2}\eta_i{}^{AB}\eta_{jCA}\eta_{DB}^k\theta^{CD}, \quad (2)$$

where $\eta_{ij}{}^k$ are the structural constants of the algebra, $\eta_i{}^{AB}$ are the connecting operators: a solution of the Clifford equation (4) [5]. According to the equation (48) [5]

$$\theta^{CD} := (\theta_0)^{CD} + (\theta_a)^{CD}, \quad (3)$$

according to the equation (46) [5]

$$(\theta_a)^{CD} = (\theta_a)^{DC} := -\frac{4}{3\sqrt{2}N}(\eta_a)_{lm}{}^r\eta^l{}_{XY}\eta^{mXC}\eta_r{}^{DY} = \frac{4}{3\sqrt{2}N}(\eta_a)_{lm}{}^r\eta^l{}_{XY}\eta^{mCX}\eta_r{}^{DY}, \quad (4)$$

according to the equation (44) [5]

$$(\theta_0)^{CD} = \frac{2}{N}\varepsilon^{CD}, \quad (5)$$

where ε^{CD} is the metric spin-tensor on the spinor space, and for $n = 8$, $N = 8$. Let the connecting operators for n=8 be given according to [4, eq. (439), p. 66(eng), eq.

(439), p. 177(88)(rus)] and the matrixes of the metric tensors have the form [4, eq. (438), p. 65(eng), eq. (438), p. 177(88)(rus)]

$$\begin{aligned}
\eta^2_{12} &= -\frac{1}{\sqrt{2}}, & \eta^2_{34} &= -\frac{1}{\sqrt{2}}, & \eta^4_{12} &= +\frac{i}{\sqrt{2}}, & \eta^4_{34} &= -\frac{i}{\sqrt{2}}, \\
\eta^2_{78} &= -\frac{1}{\sqrt{2}}, & \eta^2_{56} &= -\frac{1}{\sqrt{2}}, & \eta^4_{78} &= +\frac{i}{\sqrt{2}}, & \eta^4_{56} &= -\frac{i}{\sqrt{2}}, \\
\eta^5_{14} &= +\frac{i}{\sqrt{2}}, & \eta^5_{23} &= -\frac{i}{\sqrt{2}}, & \eta^7_{14} &= -\frac{1}{\sqrt{2}}, & \eta^7_{23} &= -\frac{1}{\sqrt{2}}, \\
\eta^5_{67} &= +\frac{i}{\sqrt{2}}, & \eta^5_{58} &= -\frac{i}{\sqrt{2}}, & \eta^7_{67} &= -\frac{1}{\sqrt{2}}, & \eta^7_{58} &= -\frac{1}{\sqrt{2}}, \\
\eta^6_{13} &= -\frac{i}{\sqrt{2}}, & \eta^6_{24} &= -\frac{i}{\sqrt{2}}, & \eta^8_{13} &= +\frac{1}{\sqrt{2}}, & \eta^8_{24} &= -\frac{1}{\sqrt{2}}, \\
\eta^6_{68} &= +\frac{i}{\sqrt{2}}, & \eta^6_{57} &= +\frac{i}{\sqrt{2}}, & \eta^8_{68} &= -\frac{1}{\sqrt{2}}, & \eta^8_{57} &= +\frac{1}{\sqrt{2}}, \\
\eta^1_{15} &= +\frac{1}{\sqrt{2}}, & \eta^1_{51} &= +\frac{1}{\sqrt{2}}, & \eta^3_{15} &= -\frac{i}{\sqrt{2}}, & \eta^3_{51} &= +\frac{i}{\sqrt{2}}, \\
\eta^1_{26} &= +\frac{1}{\sqrt{2}}, & \eta^1_{62} &= +\frac{1}{\sqrt{2}}, & \eta^3_{26} &= -\frac{i}{\sqrt{2}}, & \eta^3_{62} &= +\frac{i}{\sqrt{2}}, \\
\eta^1_{37} &= +\frac{1}{\sqrt{2}}, & \eta^1_{73} &= +\frac{1}{\sqrt{2}}, & \eta^3_{37} &= -\frac{i}{\sqrt{2}}, & \eta^3_{73} &= +\frac{i}{\sqrt{2}}, \\
\eta^1_{48} &= +\frac{1}{\sqrt{2}}, & \eta^1_{84} &= +\frac{1}{\sqrt{2}}, & \eta^3_{48} &= -\frac{i}{\sqrt{2}}, & \eta^3_{84} &= +\frac{i}{\sqrt{2}}.
\end{aligned} \tag{6}$$

We need to define the transformation

$$\frac{1}{\sqrt{2}} \begin{pmatrix} E & E \\ -iE & iE \end{pmatrix} \tag{7}$$

of the spinor basis, where E is the identity matrix 4×4 . In the new basis, the matrix of the metric spin-tensor will have the diagonal form with $\ll +1 \gg$ on the main diagonal. The matrix of the involution will have the same form. With the help of such the connecting operators, the double covering $Spin(8, \mathbb{R}) / \pm 1 \cong SO(8, \mathbb{R})$ is described as

$$S_i^j \eta_j^{AB} = \eta_i^{CD} \tilde{S}_C^A \tilde{S}_D^B, \quad \tilde{S}_C^A \tilde{S}_D^B \varepsilon_{AB} = \varepsilon_{CD}, \quad \tilde{\tilde{S}}_C^A \tilde{\tilde{S}}_D^B \varepsilon_{AB} = \varepsilon_{CD}, \tag{8}$$

according to Corollary 8.3 [3, p. 44 (eng), p. 174(51) (rus)]. If S_i^j keeps the algebra identity then $S_C^A = \tilde{S}_C^A = \tilde{\tilde{S}}_C^A$.

Example 1. Octonion [6].

Table 1: The canonical octonion multiplication table.

*	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Then in the new basis, the matrix of the symmetric spin-tensor θ^{CD} (and θ_{CD} ,

because the matrix of ε_{CD} is the identity one) has the form

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (9)$$

In the transition to the subgroup keeping the octonion identity, the dimension reduces on 7, in the transition to the subgroup keeping the symmetric spin-tensor, the dimension reduces on 7 else then the dimension of $\text{Aut}(\mathbb{A}^8)$ is equal to $\frac{8*7}{2} - 7 - 7 = 14$. Thus, we obtain the Lie group G_2 . Passing to the infinitesimal transformations according to Section 10 [3, p. 52 (eng), pp. 183-184(60-61) (rus)], we obtain

$$T_C^A = \frac{1}{2} T^{ij} \eta_j^{AB} \eta_{iCB}. \quad (10)$$

The preservation of the metrics means that

$$T_{CA} = -T_{AC}, \quad T_{ij} = -T_{ji}. \quad (11)$$

The dimension of the antisymmetrical space is equal to $\frac{8*7}{2} = 28$. In the transition to the subgroup keeping the octonion identity, we obtain

$$\eta_i^{AB} T_{AB} = 0, \quad \eta^i T_{ij} = 0. \quad (12)$$

In the new basis,

$$\begin{cases} -T_{12} - T_{34} + T_{56} + T_{78} = 0, \\ +T_{13} - T_{24} - T_{57} + T_{68} = 0, \\ -T_{14} - T_{23} + T_{58} + T_{67} = 0, \\ +T_{15} + T_{26} + T_{37} + T_{48} = 0, \\ +T_{16} - T_{25} - T_{38} + T_{47} = 0, \\ -T_{17} - T_{28} + T_{35} + T_{46} = 0, \\ +T_{18} - T_{27} + T_{36} - T_{45} = 0. \end{cases} \quad (13)$$

This reduces the dimension on 7. In the transition to the subgroup keeping the symmetric spin-tensor, we obtain

$$T_A^C \theta_{CB} + T_B^C \theta_{AC} = 0. \quad (14)$$

In the new basis,

$$T_{12} = 0, \quad T_{13} = 0, \quad T_{14} = 0, \quad T_{15} = 0, \quad T_{16} = 0, \quad T_{17} = 0, \quad T_{18} = 0. \quad (15)$$

Because the symmetric spin-tensor has the single significant eigenvalue, this reduces the dimension on 7 else. Thus, we obtain the Lie algebra g_2 whose the dimension is equal to 14. The system (13) with the relations (15) coincides with the one [8, eq. (13), p. 313] up to an orthogonal transformation.

Example 2. *The generating octonion algebra (for e_1).*

Table 2: The multiplication table of the generating octonion algebra.

*	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1				
e_3	e_3	e_2	$-e_1$	$-e_0$				
e_4	e_4	$-e_5$			$-e_0$	e_1		
e_5	e_5	e_4			$-e_1$	$-e_0$		
e_6	e_6	e_7					$-e_0$	$-e_1$
e_7	e_7	$-e_6$					e_1	$-e_0$

Then in the new basis, the matrix of the symmetric spin-tensor θ^{CD} has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (16)$$

In the new basis, the relations

$$T_{23} = 0, \quad T_{24} = 0, \quad T_{25} = 0, \quad T_{26} = 0, \quad T_{27} = 0, \quad T_{28} = 0 \quad (17)$$

are added to the ones from (15). Because the symmetric spin-tensor has the two significant eigenvalue, this reduces the dimension on 6 else. However, we need to remove the relation $T_{12} = 0$ from (15). Thus, we obtain the Lie algebra whose the dimension is equal to $14-6+1=9$.

Example 3. *The quaternion algebra analog.*

Table 3: The multiplication table of the quaternion algebra analog.

*	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$				
e_2	e_2	$-e_3$	$-e_0$	e_1				
e_3	e_3	e_2	$-e_1$	$-e_0$				
e_4	e_4				$-e_0$			
e_5	e_5					$-e_0$		
e_6	e_6						$-e_0$	
e_7	e_7							$-e_0$

Then in the new basis, the matrix of the symmetric spin-tensor θ^{CD} has the form

$$\begin{pmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (18)$$

In the new basis, the relations

$$T_{53} = 0, \quad T_{54} = 0, \quad T_{56} = 0, \quad T_{57} = 0, \quad T_{58} = 0 \quad (19)$$

are added to the ones from (15), (17). Because the symmetric spin-tensor has the four significant eigenvalue, this reduces the dimension on 5 else. Thus, we obtain the Lie algebra whose the dimension is equal to $8-5=3$. This means that the group $\text{Aut}(\mathbb{A}^4)$ is isomorphic to $SO(3, \mathbb{R})$. Of course, we need to append $SO(4, \mathbb{R})$ to obtain the all automorphism group.

Example 4. The carcass for the octonion algebra.

Table 4: The multiplication table of the octonion carcass.

*	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	0	0	0	0	0	0	0	0
e_1	0	0	e_3	$-e_2$	0	0	$-e_7$	e_6
e_2	0	$-e_3$	0	e_1	0	e_7	0	$-e_5$
e_3	0	e_2	$-e_1$	0	0	$-e_6$	e_5	0
e_4	0	0	0	0	0	0	0	0
e_5	0	0	$-e_7$	e_6	0	0	$-e_3$	e_2
e_6	0	e_7	0	$-e_5$	0	e_3	0	$-e_1$
e_7	0	$-e_6$	e_5	0	0	$-e_2$	e_1	0

Then in the new basis, the matrix of the symmetric spin-tensor θ^{CD} has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (20)$$

Example 5. *The generation octonion algebra (for e_4).*

Table 5: The multiplication table of the generating octonion algebra.

*	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	0	0	0	e_5	$-e_4$	0	0
e_2	e_2	0	0	0	e_6	0	$-e_4$	0
e_3	e_3	0	0	0	e_7	0	0	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	0	0	$-e_1$	0	0	0
e_6	e_6	0	e_4	0	$-e_2$	0	0	0
e_7	e_7	0	0	e_4	$-e_3$	0	0	0

Then in the new basis, the matrix of the symmetric spin-tensor θ^{CD} has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (21)$$

Example 6. *The algebra of the octonion type.*

Table 6: The octonion multiplication table (non-canonical).

*	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	e_5	$-e_4$
e_3	e_3	e_2	$-e_1$	$-e_0$	$-e_6$	$-e_7$	e_4	e_5
e_4	e_4	$-e_5$	$-e_7$	e_6	$-e_0$	e_1	$-e_3$	e_2
e_5	e_5	e_4	e_6	e_7	$-e_1$	$-e_0$	$-e_2$	$-e_3$
e_6	e_6	e_7	$-e_5$	$-e_4$	e_2	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_4	$-e_5$	$-e_2$	e_3	e_1	$-e_0$

Then in the new basis, the matrix of the symmetric spin-tensor θ^{CD} has the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (22)$$

Algorithm 1. *Thus, in order to compare the two algebras for $n=8$, it is necessary*

1. *to lead the both algebras to the same algebra identity;*
2. *to solve the Clifford equation, in order to find the connecting operators;*
3. *to construct the controlling symmetric spin-tensor on the structural constants and the connecting operators for each algebra;*
4. *to find the eigenvalues of the controlling spin-tensors.*

Then

1. *if the eigenvalues coincide then such the algebras are isomorphic;*
2. *if the eigenvalues not coincide then such the algebras are not isomorphic.*

This is really so, because any real symmetric spin-tensor can be transformed to the diagonal form by orthogonal transformations. In the other hand, the orthogonal transformation of the spinor space generates the orthogonal transformation of the base space, and hence this leads to the isomorphic algebra. Thus, really in order to compare the two algebras for $n=8$, it is sufficient to compare the eigenvalues of the controlling spin-tensors θ^{CD} .

Theorem 1. *The classification of metric hypercomplex group alternative-elastic algebras for $n=8$ can be reduced to the classification based on the eigenvalues of the symmetric controlling spin-tensor.*

Thus, all is made necessary for a technical realization.

Example 7. *Technical realization is given in Appendix.*

1. *This article contains the file "octonion.pas" (by the operator \input{octonion.pas}) being a programming unit adapted to the LaTeX (LaTeX version of this article is on arXiv) for the Delphi. At the same time, this file is Appendix to this article. You must create a project with this "unit octonion" and put on the form: Button1: TButton, ComboBox1: TComboBox, StringGrid1: TStringGrid (the lines 22-24).*
2. *At the lines 100-119, the structural constants η_{ij}^k for \mathbb{R}^8 ($i, j, k, \dots = \overline{1, 8}$) are initialized for Example 1.*
3. *At the lines 122-141, the structural constants η_{ij}^k for \mathbb{R}^8 are initialized for Example 2.*
4. *At the lines 144-163, the structural constants η_{ij}^k for \mathbb{R}^8 are initialized for Example 3.*
5. *At the lines 166-185, the structural constants η_{ij}^k for \mathbb{R}^8 are initialized for Example 4.*
6. *At the lines 188-207, the structural constants η_{ij}^k for \mathbb{R}^8 are initialized for Example 6.*

7. *At the lines 210-227, the connecting operators $\sqrt{2}\eta_i^{AB}$ for \mathbb{R}^8 ($i, A, B = \overline{1,8}$) are constructed.*
8. *At the lines 230-251, the transition to the new basis is constructed.*
9. *At the lines 252-281, the symmetric spin-tensor θ^{CD} is constructed.*
10. *At the lines 284-294, the symmetric spin-tensor is outputted.*
11. *At the lines 295-324, the structural constants are constructed on the symmetric spin-tensor with the help of the reverse motion.*
12. *At the lines 325-338, the symmetric spin-tensor is outputted into the file.*
13. *System characteristics of the computer on which the program is tested:
HP Pavilion dv7-6b50er i3-2330M/4096/500/Radeon HD6770 2Gb/Win7 HP64*
14. *Run time <1 sec.*

Appendix.

```
//
2 {*****}
3 {
4   Octonion Unit for Delphi
5 }
6 { Copyright (c) 2012 Konstantin Andreev All Rights Reserved
7   THIS SOFTWARE IS PROVIDED "AS IS" WITHOUT WARRANTY OF ANY KIND, EITHER EXPRESSED OR IMPLIED,
8   INCLUDING BUT NOT LIMITED TO THE IMPLIED MERCHANTABILITY AND/OR FITNESS FOR A PARTICULAR
9   PURPOSE. KONSTANTIN ANDREEV CANNOT BE HELD RESPONSIBLE FOR ANY LOSSES, EITHER DIRECT OR
10  INDIRECT, OF ANY PARTY MAKING USE OF THIS SOFTWARE. IN MAKING USE OF THIS SOFTWARE, YOU
11  AGREE TO BE BOUND BY THE TERMS AND CONDITIONS FOUND IN THE ACCOMPANYING LICENSE.
12 }
13 {*****}
14 unit octonion;
15
16 interface
17 uses
18   Windows, Messages, SysUtils, Classes, Graphics, Controls, Forms, Dialogs,
19   StdCtrls, Grids, IniFiles;
20 type
21   TForm1 = class(TForm)
22     Button1: TButton;
23     ComboBox1: TComboBox;
24     StringGrid1: TStringGrid;
25     procedure Button1Click(Sender: TObject);
26     procedure FormActivate(Sender: TObject);
27   end;
28 var
29   Form1: TForm1;
30 implementation
31 {$R *.DFM} //
32
33 procedure TForm1.Button1Click(Sender: TObject);
34 type
35   complex = record x,y : real; end;
36 var
37   eta_8 : array [1..8,1..8,1..8] of complex;
38   eta_8_ : array [1..8,1..8,1..8] of complex;
39   eta1 : array [1..8,1..8,1..8,1..8] of complex;
40   eta2 : array [1..8,1..8,1..8] of complex;
41   S_trans_8 : array [1..8,1..8] of complex;
42   Theta : array [1..8,1..8] of complex;
43   eta : array [1..8,1..8,1..8] of complex;
44   eta_ : array [1..8,1..8,1..8] of complex;
45   m_1 : complex;
46   m_2 : complex;
47   m_24 : complex;
48   i,j,k,l,m,r : integer;
49   A,B,C,D : integer;
50   IniFile : TIniFile;
51   array_str : string;
52
53 //The connection operators
54 //for n=8  $\eta_i^{AB}$ .
55 //The auxiliary variable.
56 //The auxiliary variable.
57 //The auxiliary variable.
58 //The orthogonal transformation (7) to
59 //the new basis.
60 //The matrix of the quadratic form.
61 //||  $\theta_{AB}$  ||= $\theta^{AB}$  ||.
62 //The structural constants of
63 //the algebra for n=8  $\eta_{ij}^k$ .
64 //The auxiliary variable.
65 //The complex factor -1.
66 //The complex factor  $\frac{1}{2}$ .
67 //The complex factor  $\frac{1}{24}$ .
68 //Indices of an array element.
69 //Indices of an array element.
70 //The output file.
71 //A line of the output file.
72
73 //Addition of complex numbers.
74 function add(c11,c12: complex):complex;
75 var c13 : complex;
76 begin
77   c13.x:=c11.x+c12.x;
78   c13.y:=c11.y+c12.y;
79   add:=c13;
80 end;
81 //Multiplication of complex numbers.
82 function mul(c21,c22: complex):complex;
83 var c13 : complex;
84 begin
85   c13.x:=c21.x*c22.x-c21.y*c22.y;
86   c13.y:=c21.x*c22.y+c21.y*c22.x;
87   mul:=c13;
88 end;
89 //Initialization of a complex number.
90 procedure Init_(var c31 : complex);
91 begin
92   c31.x:=0; c31.y:=0;
93 end;
94
95 begin
96   //Initialization of the connection operators for n=8.
97   for i:=1 to 8 do
98     for j:=1 to 8 do
99       for k:=1 to 8 do
100         begin
101           Init_(eta_8[i,j,k]);
102           Init_(eta1[i,j,k]);
103         end;
104       end;
105     end;
106   //Initialization of  $\Theta^{CD}$ .
107   for C:=1 to 8 do
108     for D:=1 to 8 do
109       Init_(Theta[C,D]);
110     end;
111   end;
112 end;
```

```

92 //Initialization of the base transformation.
93   for C:=1 to 8 do
94     for D:=1 to 8 do
95       Init_(S_trans_8[C,D]);
96 //Initialization of the structural constants of the algebra:
97 Case ComboBox1.ItemIndex of
98
99
100 0:begin
101 //Octonion.
102   eta[1,1,1].x:=0; eta[1,2,2].x:= 0; eta[1,3,3].x:= 0; eta[1,4,4].x:= 0;
103   eta[2,1,2].x:=0; eta[2,2,1].x:= 0; eta[2,3,4].x:=+1; eta[2,4,3].x:=-1;
104   eta[3,1,3].x:=0; eta[3,2,4].x:=-1; eta[3,3,1].x:= 0; eta[3,4,2].x:=+1;
105   eta[4,1,4].x:=0; eta[4,2,3].x:=+1; eta[4,3,2].x:=-1; eta[4,4,1].x:= 0;
106   eta[5,1,5].x:=0; eta[5,2,6].x:=-1; eta[5,3,7].x:=-1; eta[5,4,8].x:=-1;
107   eta[6,1,6].x:=0; eta[6,2,5].x:=+1; eta[6,3,8].x:=-1; eta[6,4,7].x:=+1;
108   eta[7,1,7].x:=0; eta[7,2,8].x:=+1; eta[7,3,5].x:=+1; eta[7,4,6].x:=-1;
109   eta[8,1,8].x:=0; eta[8,2,7].x:=-1; eta[8,3,6].x:=+1; eta[8,4,5].x:=+1;
110
111   eta[1,5,5].x:= 0; eta[1,6,6].x:= 0; eta[1,7,7].x:= 0; eta[1,8,8].x:= 0;
112   eta[2,5,6].x:=+1; eta[2,6,5].x:=-1; eta[2,7,8].x:=-1; eta[2,8,7].x:=+1;
113   eta[3,5,7].x:=+1; eta[3,6,8].x:=+1; eta[3,7,5].x:=-1; eta[3,8,6].x:=-1;
114   eta[4,5,8].x:=+1; eta[4,6,7].x:=-1; eta[4,7,6].x:=+1; eta[4,8,5].x:=-1;
115   eta[5,5,1].x:= 0; eta[5,6,2].x:=+1; eta[5,7,3].x:=+1; eta[5,8,4].x:=+1;
116   eta[6,5,2].x:=-1; eta[6,6,1].x:= 0; eta[6,7,4].x:=-1; eta[6,8,3].x:=+1;
117   eta[7,5,3].x:=-1; eta[7,6,4].x:=+1; eta[7,7,1].x:= 0; eta[7,8,2].x:=-1;
118   eta[8,5,4].x:=-1; eta[8,6,3].x:=-1; eta[8,7,2].x:=+1; eta[8,8,1].x:= 0;
119 end;
120
121
122 1:begin
123 //The generating octonion algebra.
124   eta[1,1,1].x:=0; eta[1,2,2].x:= 0; eta[1,3,3].x:= 0; eta[1,4,4].x:= 0;
125   eta[2,1,2].x:=0; eta[2,2,1].x:= 0; eta[2,3,4].x:=+1; eta[2,4,3].x:=-1;
126   eta[3,1,3].x:=0; eta[3,2,4].x:=-1; eta[3,3,1].x:= 0; eta[3,4,2].x:=+1;
127   eta[4,1,4].x:=0; eta[4,2,3].x:=+1; eta[4,3,2].x:=-1; eta[4,4,1].x:= 0;
128   eta[5,1,5].x:=0; eta[5,2,6].x:=-1;
129   eta[6,1,6].x:=0; eta[6,2,5].x:=+1;
130   eta[7,1,7].x:=0; eta[7,2,8].x:=+1;
131   eta[8,1,8].x:=0; eta[8,2,7].x:=-1;
132
133   eta[1,5,5].x:= 0; eta[1,6,6].x:= 0; eta[1,7,7].x:= 0; eta[1,8,8].x:= 0;
134   eta[2,5,6].x:=+1; eta[2,6,5].x:=-1; eta[2,7,8].x:=-1; eta[2,8,7].x:=+1;
135
136
137   eta[5,5,1].x:= 0; eta[5,6,2].x:=+1;
138   eta[6,5,2].x:=-1; eta[6,6,1].x:= 0;
139   eta[7,7,1].x:= 0; eta[7,8,2].x:=-1;
140   eta[8,7,2].x:=+1; eta[8,8,1].x:= 0;
141 end;
142
143
144 2:begin
145 //The quaternion algebra analog.
146   eta[1,1,1].x:=0; eta[1,2,2].x:= 0; eta[1,3,3].x:= 0; eta[1,4,4].x:= 0;
147   eta[2,1,2].x:=0; eta[2,2,1].x:= 0; eta[2,3,4].x:=+1; eta[2,4,3].x:=-1;
148   eta[3,1,3].x:=0; eta[3,2,4].x:=-1; eta[3,3,1].x:= 0; eta[3,4,2].x:=+1;
149   eta[4,1,4].x:=0; eta[4,2,3].x:=+1; eta[4,3,2].x:=-1; eta[4,4,1].x:= 0;
150   eta[5,1,5].x:=0;
151   eta[6,1,6].x:=0;
152   eta[7,1,7].x:=0;
153   eta[8,1,8].x:=0;
154
155   eta[1,5,5].x:= 0; eta[1,6,6].x:= 0; eta[1,7,7].x:= 0; eta[1,8,8].x:= 0;
156
157
158
159   eta[5,5,1].x:= 0;
160   eta[6,6,1].x:= 0;
161   eta[7,7,1].x:= 0;
162   eta[8,8,1].x:= 0;
163 end;
164
165
166 3:begin
167 //The carcaess for the octonion algebra.
168   eta[1,1,1].x:=0; eta[1,2,2].x:= 0; eta[1,3,3].x:= 0; eta[1,4,4].x:= 0;
169   eta[2,1,2].x:=0; eta[2,2,1].x:= 0; eta[2,3,4].x:=+1; eta[2,4,3].x:=-1;
170   eta[3,1,3].x:=0; eta[3,2,4].x:=-1; eta[3,3,1].x:= 0; eta[3,4,2].x:=+1;
171   eta[4,1,4].x:=0; eta[4,2,3].x:=+1; eta[4,3,2].x:=-1; eta[4,4,1].x:= 0;
172   eta[5,1,5].x:=0; eta[5,2,6].x:= 0; eta[5,3,7].x:= 0; eta[5,4,8].x:= 0;
173   eta[6,1,6].x:=0; eta[6,2,5].x:= 0; eta[6,3,8].x:=-1; eta[6,4,7].x:=+1;
174   eta[7,1,7].x:=0; eta[7,2,8].x:=+1; eta[7,3,5].x:= 0; eta[7,4,6].x:=-1;
175   eta[8,1,8].x:=0; eta[8,2,7].x:=-1; eta[8,3,6].x:=+1; eta[8,4,5].x:= 0;
176
177   eta[1,5,5].x:= 0; eta[1,6,6].x:= 0; eta[1,7,7].x:= 0; eta[1,8,8].x:= 0;
178   eta[2,5,6].x:= 0; eta[2,6,5].x:= 0; eta[2,7,8].x:=-1; eta[2,8,7].x:=+1;
179   eta[3,5,7].x:= 0; eta[3,6,8].x:=+1; eta[3,7,5].x:= 0; eta[3,8,6].x:=-1;
180   eta[4,5,8].x:= 0; eta[4,6,7].x:=-1; eta[4,7,6].x:=+1; eta[4,8,5].x:= 0;
181   eta[5,5,1].x:= 0; eta[5,6,2].x:= 0; eta[5,7,3].x:= 0; eta[5,8,4].x:= 0;
182   eta[6,5,2].x:= 0; eta[6,6,1].x:= 0; eta[6,7,4].x:=-1; eta[6,8,3].x:=+1;
183   eta[7,5,3].x:= 0; eta[7,6,4].x:=+1; eta[7,7,1].x:= 0; eta[7,8,2].x:=-1;
184   eta[8,5,4].x:= 0; eta[8,6,3].x:=-1; eta[8,7,2].x:=+1; eta[8,8,1].x:= 0;
185 end;
186
187

```

```

188 4:begin
189 //The octonion algebra analog.
190 eta[1,1,1].x:=0; eta[1,2,2].x:= 0; eta[1,3,3].x:= 0; eta[1,4,4].x:= 0;
191 eta[2,1,2].x:=0; eta[2,2,1].x:= 0; eta[2,3,4].x:=+1; eta[2,4,3].x:=+1;
192 eta[3,1,3].x:=0; eta[3,2,4].x:=+1; eta[3,3,1].x:= 0; eta[3,4,2].x:=+1;
193 eta[4,1,4].x:=0; eta[4,2,3].x:=+1; eta[4,3,2].x:=+1; eta[4,4,1].x:= 0;
194 eta[5,1,5].x:=0; eta[5,2,6].x:=+1; eta[5,3,8].x:=+1; eta[5,4,7].x:=+1;
195 eta[6,1,6].x:=0; eta[6,2,5].x:=+1; eta[6,3,7].x:=+1; eta[6,4,8].x:=+1;
196 eta[7,1,7].x:=0; eta[7,2,8].x:=+1; eta[7,3,6].x:=+1; eta[7,4,5].x:=+1;
197 eta[8,1,8].x:=0; eta[8,2,7].x:=+1; eta[8,3,5].x:=+1; eta[8,4,6].x:=+1;
198
199 eta[1,5,5].x:= 0; eta[1,6,6].x:= 0; eta[1,7,7].x:= 0; eta[1,8,8].x:= 0;
200 eta[2,5,6].x:=+1; eta[2,6,5].x:=+1; eta[2,7,8].x:=+1; eta[2,8,7].x:=+1;
201 eta[3,5,8].x:=+1; eta[3,6,7].x:=+1; eta[3,8,5].x:=+1; eta[3,7,6].x:=+1;
202 eta[4,5,7].x:=+1; eta[4,6,8].x:=+1; eta[4,7,5].x:=+1; eta[4,8,6].x:=+1;
203 eta[5,5,1].x:= 0; eta[5,6,2].x:=+1; eta[5,7,4].x:=+1; eta[5,8,3].x:=+1;
204 eta[6,5,2].x:=+1; eta[6,6,1].x:= 0; eta[6,7,3].x:=+1; eta[6,8,4].x:=+1;
205 eta[7,6,3].x:=+1; eta[7,5,4].x:=+1; eta[7,7,1].x:= 0; eta[7,8,2].x:=+1;
206 eta[8,5,3].x:=+1; eta[8,6,4].x:=+1; eta[8,7,2].x:=+1; eta[8,8,1].x:= 0;
207 end;
208 end;
209
210 //Construction of the connection operators for n=8 (multiplied by  $\sqrt{2}$ ).
211 eta_8[2,1,2].x:=+1; eta_8[2,3,4].x:=+1; eta_8[2,5,6].x:=+1; eta_8[2,7,8].x:=+1;
212 eta_8[4,1,2].y:=+1; eta_8[4,3,4].y:=+1; eta_8[4,5,6].y:=+1; eta_8[4,7,8].y:=+1;
213 eta_8[5,1,4].y:=+1; eta_8[5,2,3].y:=+1; eta_8[5,5,8].y:=+1; eta_8[5,6,7].y:=+1;
214 eta_8[7,1,4].x:=+1; eta_8[7,2,3].x:=+1; eta_8[7,5,8].x:=+1; eta_8[7,6,7].x:=+1;
215 eta_8[6,1,3].y:=+1; eta_8[6,2,4].y:=+1; eta_8[6,5,7].y:=+1; eta_8[6,6,8].y:=+1;
216 eta_8[8,1,3].x:=+1; eta_8[8,2,4].x:=+1; eta_8[8,5,7].x:=+1; eta_8[8,6,8].x:=+1;
217 for i:=1 to 8 do
218   for j:=1 to 8 do
219     for k:=1 to j do
220       begin
221         eta_8[i,j,k].x:=+eta_8[i,k,j].x;
222         eta_8[i,j,k].y:=+eta_8[i,k,j].y;
223       end;
224     eta_8[3,1,5].y:=+1; eta_8[3,2,6].y:=+1; eta_8[3,3,7].y:=+1; eta_8[3,4,8].y:=+1;
225     eta_8[3,5,1].y:=+1; eta_8[3,6,2].y:=+1; eta_8[3,7,3].y:=+1; eta_8[3,8,4].y:=+1;
226     eta_8[1,1,5].x:=+1; eta_8[1,2,6].x:=+1; eta_8[1,3,7].x:=+1; eta_8[1,4,8].x:=+1;
227     eta_8[1,5,1].x:=+1; eta_8[1,6,2].x:=+1; eta_8[1,7,3].x:=+1; eta_8[1,8,4].x:=+1;
228 //The constant factors:  $1/2(m-2), 1/24(m-24), -1(m-1)$ .
229 m_2.x:=1/2; m_2.y:=0; m_24.x:=1/24; m_24.y:=0; m_1.x:=+1; m_1.y:=0;
230 //Transition to the new spinor basis.
231 S_trans_8[1,5].x:=+1; S_trans_8[2,6].x:=+1; S_trans_8[3,7].x:=+1; S_trans_8[4,8].x:=+1;
232 S_trans_8[1,1].x:=+1; S_trans_8[2,2].x:=+1; S_trans_8[3,3].x:=+1; S_trans_8[4,4].x:=+1;
233 S_trans_8[5,1].y:=+1; S_trans_8[6,2].y:=+1; S_trans_8[7,3].y:=+1; S_trans_8[8,4].y:=+1;
234 S_trans_8[5,5].y:=+1; S_trans_8[6,6].y:=+1; S_trans_8[7,7].y:=+1; S_trans_8[8,8].y:=+1;
235 for i:=1 to 8 do
236   for A:=1 to 8 do
237     for B:=1 to 8 do
238       begin
239         Init_(eta_8[i,A,B]);
240         for C:=1 to 8 do
241           eta_8[i,A,B]:=add(eta_8[i,A,B],mul(eta_8[i,A,C],S_trans_8[B,C]));
242         end;
243       for i:=1 to 8 do
244         for A:=1 to 8 do
245           for B:=1 to 8 do
246             begin
247               Init_(eta_8[i,A,B]);
248               for D:=1 to 8 do
249                 eta_8[i,A,B]:=add(eta_8[i,A,B],mul(eta_8[i,D,B],S_trans_8[A,D]));
250                 eta_8[i,A,B]:=mul(eta_8[i,A,B],m_2);
251               end;
252             //Construction of the controlling spin-tensor according to the equation (3)-(5).
253             for j:=1 to 8 do
254               for k:=1 to 8 do
255                 for A:=1 to 8 do
256                   for B:=1 to 8 do
257                     begin
258                       Init_(eta1[j,k,A,B]);
259                       for i:=1 to 8 do
260                         eta1[j,k,A,B]:=add(eta1[j,k,A,B],mul(eta[i,j,k],eta_8[i,A,B]));
261                       end;
262                     for k:=1 to 8 do
263                       for B:=1 to 8 do
264                         for C:=1 to 8 do
265                           begin
266                             Init_(eta2[k,B,C]);
267                             for j:=1 to 8 do
268                               for A:=1 to 8 do
269                                 eta2[k,B,C]:=add(eta2[k,B,C],mul(eta1[j,k,A,B],eta_8[j,C,A]));
270                               end;
271                             for C:=1 to 8 do
272                               for D:=1 to 8 do
273                                 begin
274                                   Init_(Theta[B,D]);
275                                   for k:=1 to 8 do
276                                     for B:=1 to 8 do
277                                       Theta[C,D]:=add(Theta[C,D],mul(eta2[k,B,C],eta_8[k,D,B]));
278                                       Theta[C,D]:=mul(Theta[C,D],m_24);
279                                       if ComboBox1.ItemIndex<>3 then
280                                         if (C=D) then Theta[C,D].x:=Theta[C,D].x+1/4;
281                                       end;
282                                     end;
283                                   end;
284                                 end;
285                               end;
286                             end;
287                           end;
288                         end;
289                       end;
290                     end;
291                   end;
292                 end;
293               end;
294             end;
295           end;
296         end;
297       end;
298     end;
299   end;
300 end;

```

```

284 //Output of the  $\Theta$ -matrix.
285 for i:=1 to 8 do
286   begin
287     StringGrid1.Cells[0,i]:=IntToStr(i);
288     StringGrid1.Cells[i,0]:=IntToStr(i);
289   end;
290   for i:=1 to 8 do
291     for j:=1 to 8 do
292       begin
293         StringGrid1.Cells[j,i]:=FloatToStr(Theta[i,j].x);
294       end;
295     //Reverse motion.
296     for j:=1 to 8 do
297       for C:=1 to 8 do
298         for i:=1 to 8 do
299           for B:=1 to 8 do
300             begin
301               Init_(eta1[j,C,i,B]);
302               for A:=1 to 8 do
303                 eta1[j,C,i,B]:=add(eta1[j,C,i,B],mul(eta_8[j,C,A],eta_8[i,A,B]));
304             end;
305           for k:=1 to 8 do
306             for B:=1 to 8 do
307               for C:=1 to 8 do
308                 begin
309                   Init_(eta2[k,B,C]);
310                   for D:=1 to 8 do
311                     eta2[k,B,C]:=add(eta2[k,B,C],mul(eta_8[k,D,B],Theta[C,D]));
312                 end;
313               end;
314             for i:=1 to 8 do
315               for j:=1 to 8 do
316                 for k:=1 to 8 do
317                   begin
318                     Init_(eta_[i,j,k]);
319                     for B:=1 to 8 do
320                       for C:=1 to 8 do
321                         eta_[i,j,k]:=add(eta_[i,j,k],mul(eta2[k,B,C],eta1[j,C,i,B]));
322                       eta_[i,j,k]:=mul(m_2,eta_[i,j,k]);
323                     end;
324                   end;
325                 //Output of the table into the file.
326                 IniFile:=TIniFile.Create(GetCurrentDir+'\IniFile.ini');
327                 array_str:='';
328                 for A:=1 to 8 do
329                   begin
330                     array_str:='';
331                     for B:=1 to 8 do
332                       begin
333                         array_str:=array_str+Format('%6.3f',[Theta[A,B].x]);
334                         if B<8 then
335                           array_str:=array_str+'_&omega_';
336                       end;
337                     IniFile.WriteString('Variant_'+IntToStr(A),array_str+'\\');
338                   end;
339                 end;
340               end;
341             procedure TForm1.FormActivate(Sender: TObject);
342             begin
343               //Initialization of ComboBox1.
344               ComboBox1.Items.Add('Octonoion');
345               ComboBox1.Items.Add('Generating_octonion_algebra');
346               ComboBox1.Items.Add('Quaternion_algebra_analog');
347               ComboBox1.Items.Add('Carcacss_for_the_octonion_algebra');
348               ComboBox1.Items.Add('Octonoion_algebra_analog');
349               ComboBox1.ItemIndex:=0;
350               StringGrid1.RowCount:=9;
351               StringGrid1.ColCount:=9;
352               StringGrid1.Align:=alBottom;
353               StringGrid1.Height:=250;
354               StringGrid1.DefaultColWidth:=30;
355             end;
356           end.
357         end.
358       //

```

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